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Plasma Simulation as Eigenvalue Problem*

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A numerical integration of the Vlasov equation replaces the continuous eigenvalue spectrum of the problem by a discrete spectrum, which can be represented by the eigenvalue spectrum of a finite matrix. This matrix must be dissipative, i.e., the eigenvalues must have a negative real part in order to avoid recurrence effects. Several dissipative terms in the diagonal of the matrix are studied numerically and analytically and the influence of various parameters is investigated. It is shown that a decrease of the last few off-diagonal terms in the matrix may also enhance the damping.

I. INTRODUCTION

In a recent paper [1] a method has been suggested how to integrate numerically the Vlasov equation and to avoid the usual recurrence effects. The recurrence effects either force one to inflate unduly the computational effort or limit seriously the time during which the numerical solution can be considered to represent a solution of the Vlasov equation. The mathematical origin of the difficulty is the simulation of a continuous eigenvalue problem [2] with a finite computer. Any kind of truncation due to the finiteness of the computer changes the problem into an eigenvalue problem with a set of finite discrete eigenvalues which are purely imaginary [3]. Consequently the solution of the truncated equations turns out to be almost periodic in the density contrary to the solutions of the original problem which are damped in the density.

The problem was transformed in Ref. [1] into the form

$$(d/dt)\, a + \mathbf{R} \cdot a = 0$$

where R is a $(M + 1) \times (M + 1)$ trigonal matrix with purely imaginary eigenvalues. The suggested remedy was to add a real part to the eigenvalues which damped the solutions by a judiciously chosen cutoff procedure which makes the terms in the last line of the matrix nonzero. This procedure proved to be quite

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successful. However, there were certain numerical stability problems associated with the addition of these terms. The main feature of the method was to make the eigenvalues complex rather than purely imaginary. It was felt that the same objectives could be obtained by adding certain diagonal terms to the matrix which also make the matrix dissipative. Such dissipative terms are closely related to certain "collision" terms and some have been investigated earlier (see, e.g. [4] and [5]).

In Section II we discuss the influence of diagonal terms on the eigenvalues of the matrix **R**. In Section III we study how damping is enhanced by varying terms in the off-diagonals. This is followed by a summary in Section IV.

II. Damping by Diagonal Terms

We study the free streaming of the Vlasov equation to which we added a "collision term"

$$(\partial f/\partial t) + v(\partial f/\partial x) = \sigma(\partial^{2m}/\partial x^{2m})\, C^{2r+1} f \tag{1}$$

where

$$C = (\partial/\partial v)((\partial/\partial v) + v). \tag{2}$$

The collision term C is discussed for Hermite polynomials by Joyce, Knorr, and Meier [3]. The case $r = 0$, $m = 0$ has been thoroughly studied by Denavit, Doyle, and Hirsch [4]. A set of eigenfunctions has been given by Lafleur [5]. For numerical simulation the case $m = 1$ appears to be advantageous because the damping of the higher k-modes is enhanced that way.

Using a Fourier series in x and a series of Hermite polynomials in v, we obtain a set of equations similar to Ref. [1] which is cut off after the M-th equation.

$$(\partial/\partial t)\, b_\nu + ik\rho_\nu(b_{\nu-1} + b_{\nu+1}) = -\sigma k^{2m}\nu^{2r+1} b_\nu\,; \qquad \nu = 0, 1,..., M. \tag{3}$$

The coefficients ρ_ν are defined by the recurrence formula

$$\rho_\nu \rho_{\nu+1} = \nu + 1, \tag{4}$$

with ρ_0 positive and arbitrary. It is seen that for increasing r preferentially the b_ν with large ν are damped. This means that preferentially the rapid fluctuations in velocity space are suppressed. Introducing $b_\nu = i^\nu a_\nu$ makes all coefficients real. We obtain

$$(d/dt)\, a + \mathbf{R} \cdot a = 0. \tag{5}$$

R is a $(M + 1) \times (M + 1)$ matrix and $R_{jj} = \sigma k^2 j^{2r+1}$ for $m = 1$;
$R_{j,j-1} = R_{j,j+1} = k\rho_j$. Equation (5) defines our eigenvalue problem.

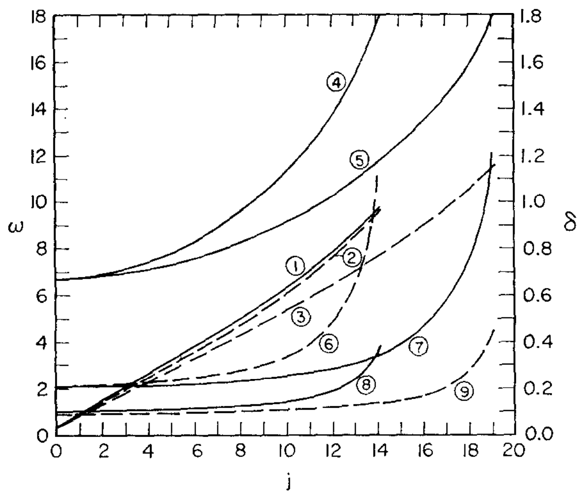

FIG. 1. Imaginary and real part of the eigenvalues $\Lambda = i\omega - \delta$ of the matrix **R** in Eq. (5) for different parameters M, $\lambda = \sigma N^{(2r+1)}$, and r: (1) ω for $M = 30$, $\lambda = 2$, $r = 0 - 8$; (2) ω for $M = 30$, $\lambda = 6$, $r = 0$; (3) ω for $M = 40$, $\lambda = 2$, $r = 0 - 8$; (4) δ for $M = 30$, $\lambda = 2$, $r = 0$; (5) δ for $M = 40$, $\lambda = 2$, $r = 0$; (6) δ for $M = 30$, $\lambda = 2$, $r = 2$; (7) δ for $M = 40$, $\lambda = 2$, $r = 2$; (8) δ for $M = 30$, $\lambda = 2$, $r = 8$; (9) δ for $M = 40$, $\lambda = 2$, $r = 8$.

The eigenvalues of Eq. (5) are easily obtained numerically by putting $b_\nu = b_\nu \exp(\Lambda t)$. We write $\Lambda = i\omega - \delta$ and plot ω and δ for various parameters in Fig. 1. $(M + 1)$ is the dimension of the matrix, λ is the magnitude of the last diagonal term, i.e., $R_{MM} = \lambda$ or $\lambda = \sigma M^{(2r+1)}$, r characterizes the power of the "collision term" C, and k is normalized to $k \equiv 1$ for all cases.

As can be seen from Fig. 1 the various roots of ω for one parameter set lie on a curve which is almost a straight line, i.e., we can write to a good approximation a solution of Eq. (3) as

$$b_\nu = \sum_{j=0}^{M} C_{\nu j}[\exp i(\epsilon + j\Delta\omega)t - \delta_j t]. \tag{6}$$

This means that for zero damping we have almost complete recurrence. Taking up the physical analogies of Ref. [1] we can also say that the disturbance travels down the ν-axis with almost no dispersion. At recurrence time

$$\tau_{\text{Rec}} = 2\pi/\Delta\omega,$$

the initial condition is recovered except for a common phase factor. This is shown in a model calculation of Eq. (3) in Fig. 2 for different damping terms.

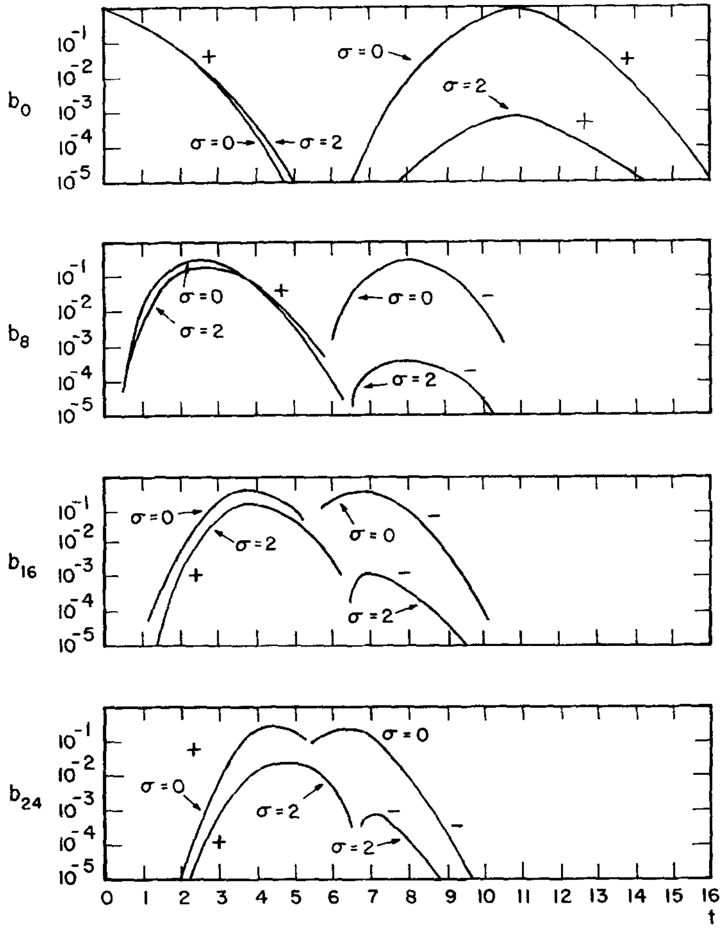

FIG. 2. Numerical solutions of Eq. (3) for $\nu = 0$, 8, 16, and 24. It is seen that the solutions exhibit very little dispersion, are reflected at the cutoff at $M = 30$, and exhibit various damping according to Eq. (8), as they travel along. The curves are calculated for $r = 0$, and the absolute value is plotted against time. The algebraic value is indicated on the figure (+ for positive, − for negative).

The asymptotic behavior of the Hermite polynomials for large M and $y \ll M$ is given by

$$He_M(y) \propto \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (yM^{1/2}) \begin{Bmatrix} M \text{ odd}, \\ M \text{ even}. \end{Bmatrix} \tag{7}$$

It follows that the slopes of the ω curves are equal to $\pi/M^{1/2}$ which is consistent with Fig. 1. The curves are almost independent of the parameter r which characterizes the power of the collision term. It is interesting to note that the deviation of the ω-curves from a straight line occurs mostly for larger j where the asymptotic condition $y \ll M^{1/2}$ is no longer well satisfied. On the other hand the damping decrements are larger for these modes for any r so that the inclusion of damping rather reduces dispersion. In addition, it can be shown that the amplitudes of the terms in Eq. (6) with stronger damping are small from the outset if gaussians or derivatives of gaussians are chosen as initial conditions.

The curves for the damping decrement δ show always an increase of damping for the modes with increasing ω. As r is increased the overall magnitude of δ goes down markedly. The minimum damping δ_0 (for $j = 0$) is proportional to λ.

The dependence of the minimum damping on the various parameters can be obtained by the following qualitative consideration: The damping of the amplitude of a disturbance as it travels down the ν-axis can be described by the factor

$$\exp\left[-\int_0^t \lambda(\nu)\, dt\right]. \tag{8}$$

According to Ref. [1] the relation between time t and position ν of the disturbance is given by

$$t = \nu^{1/2} \tag{9}$$

for Hermite polynomials. With Eq. (9) and the right side of Eq. (1) it follows that

$$\lambda(\nu) = \sigma\nu^{2r+1} = \lambda(\nu/M)^{2r+1}.$$

The integral (8) can be trivially evaluated for the case that the disturbance has traveled to the cutoff of the matrix. For the time of recurrence τ_{Rec} the amplitude is given by

$$A(\tau_{\text{Rec}}) = A_0 \exp[-2\lambda M^{1/2}/(4r + 3)]. \tag{10}$$

This relation is well satisfied for the disturbances drawn in Fig. 2.

In order to express the minimum damping decrements plotted in Fig. 2 with the aid of the result (10) we write

$$\delta_0\, \tau_{\text{Rec}} = 2\lambda M^{1/2}/(4r + 3) \tag{11}$$

because the least damped component in Eq. (6) have the largest amplitudes $C_{\nu j}$. With $\tau_{\text{Rec}} = 2M^{1/2}$ we obtain

$$\delta_0 = \lambda/(4r + 3). \tag{12}$$

The independence of δ_0 from M is conspicuous, and the relation is well satisfied. In any case we expect only qualitative agreement because we have made use of asymptotic formulas which are only approximately satisfied.

III. Off-Diagonal Damping

In the previous section it has been demonstrated that recurrence effects can be damped effectively by adding diagonal terms to the matrix **R** of Eq. (5). The question arises if the damping can be enhanced by changing the nondiagonal terms. Naturally a modification of the terms $R_{i,i\pm 1}$ are preferable because this does not involve any additional computational efforts in a numerical solution of the Vlasov equation. The argument why an increase of the damping should be possible is the following: According to Eq. (9), the speed of a disturbance is given by

$$d\nu/dt = 2\nu^{1/2}.$$

On the other hand it follows from Eq. (4) that asymptotically $\rho_\nu \propto \nu^{1/2}$. Thus the speed of propagation of a disturbance is proportional to the coefficient ρ_ν in Eq. (3). If one modified the coefficients in the last few equations before the cutoff the disturbance should dwell in that area of the matrix longer. On the other hand this is also the area of the largest damping as is evident from the right side of Eq. (3), and the disturbance experiences a stronger damping.

We have decreased the last 5 coefficients below their value given in Eq. (3) according to

$$\tilde{\rho}_\nu = \rho_\nu\{1 - p\exp[-\beta(M-\nu)]\}, \qquad M-4 \leqslant \nu \leqslant M. \tag{13}$$

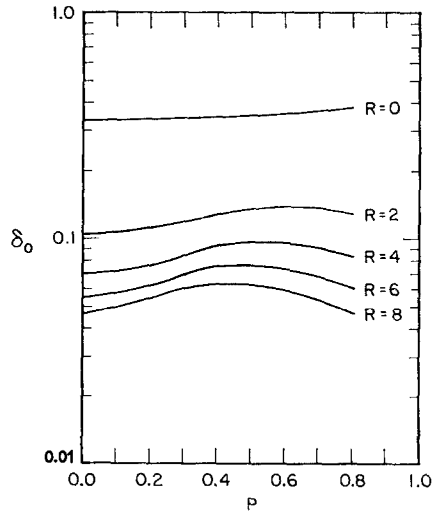

Fig. 3. The modification of the coefficients of the matrix **R** according to $\tilde{\rho}_\nu = \rho_\nu[1 - p \exp - (M-\nu)]$ for $M - 4 < \nu < M$ influences the minimum real part δ_0 of the eigenvalues of **R**. In general the damping is enhanced by the modification.

Figure 3 shows the minimum damping δ_0 as a function of p for $\beta = 1$. It is seen that the minimum damping decrement goes through a maximum as p increases. The larger r, the sooner the maximum occurs. The damping decrement can thus be increased by as much as 30 %. If β is decreased the decline in the curves is more emphasized. This indicates that too sudden a transition of the magnitude of the coefficients rather decreases than enhances damping.

IV. Summary

The study of the free streaming term of the Vlasov equation can be reduced to the study of the eigenvalue spectrum of a finite matrix which replaces the infinite matrix of the original problem. Several methods are suggested and studied to make the matrix dissipative by adding diagonal terms. These terms correspond to generalized Fokker–Planck collision terms in the Vlasov equation. An asymptotic formula is derived which allows to estimate the effectiveness of a given set of parameters.

Finally, it is shown that certain modifications in the terms of the off-diagonals may increase the damping decrement by as much as 30 %. The results of this report are being used to construct a two-dimensional plasma simulation code.

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